

Elastic properties of small-world spring networks

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We construct small-world spring networks based on a one-dimensional chain and study its static and quasistatic behavior with respect to external forces. Regular bonds and shortcuts are assigned linear springs of constant k and k' , respectively. In our models, shortcuts can only stand extensions less than δ_c beyond which they are removed from the network. First we consider the simple cases of a hierarchical small-world network and a complete network. In the main part of this paper we study random small-world networks (RSWN) in which each pair of nodes is connected by a shortcut with probability p . We obtain a scaling relation for the effective stiffness of RSWN when $k=k'$. In this case the extension distribution of shortcuts is scale free with the exponent -2 . There is a strong positive correlation between the extension of shortcuts and their betweenness. We find that the chemical end-to-end distance (CEED) could change either abruptly or continuously with respect to the external force. In the former case, the critical force is determined by the average number of shortcuts emanating from a node. In the latter case, the distribution of changes in CEED obeys power laws of the exponent $-\alpha$ with $\alpha \leq \frac{3}{2}$.

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I. INTRODUCTION

Recent studies indicate that real networks have a complex structure and function [1–5]. The small-world network (SWN) introduced by Watts and Strogatz [3] captures some basic ingredients of real networks. There are lots of studies dealing with static properties of SWNs, see [2], and references therein. However, we still only know a little bit about dynamical features of small-world networks [6–11]. To best of our knowledge there is no study on the elastic properties of SWNs. Certainly the most physical application of small-world networks is in the context of macromolecules and polymers [12–16]. The interaction pattern of a polymer can be represented by a SWN. In this way, the monomers are mapped into nodes of a network and the interaction between two spatially close monomers is shown by a bond between the corresponding nodes. Note that here there are two kinds of bonds: regular and long range. The long range bonds, that are usually weaker than the regular ones, could stand much smaller stresses and would be broken more easily. If we model the interactions with linear springs, we will obtain a small-world spring network. Of course, we should take more care on modeling linear polymers with SWNs [17]. Nevertheless, we expect that the study of small-world networks (as a toy model of polymers when thermal activities are absent) provides insights about more complicated behaviors of real polymers.

Suppose that we have a chain of elastic fibers and add some other fibers to randomly selected pairs of nodes. Then it is interesting to know, for example, the effective stiffness of this object and some other quantities which are of interest in fiber-bundle models [18,19]. In this paper we are going to study the behavior of small-world spring networks when external force F is exerted on the end nodes. The network

response contains some information about its internal structure. For a given F , we obtain the network stiffness and the extension distribution of long range bonds (shortcuts). Then we increase F quasistatically and define a cutoff length for shortcuts beyond which we remove them from the network. During this quasistatic process the chemical end-to-end distance (CEED), defined as the number of bonds in the shortest path connecting node 1 to N , could have nontrivial behavior with F . Moreover by increasing F one encounters a number of avalanches in which one has a change in the number of shortcuts and also in CEED. Certainly, distinct structures could lead to different behaviors, for instance, in the distribution of changes in CEED. These differences provide us a useful measure to classify various networks.

Here we will show that the extension distribution is scale free in the small-world regime in contrast to the three-modal distribution of complete-network regime. We obtain a positive correlation between the betweenness [2] of a shortcut and its extension. There is also a threshold value F_c which leads to an abrupt change in CEED. The scaling of F_c with the size of the network depends strongly on the network structure. Depending on the elastic properties of springs we could also have a continuous change in CEED.

The structure of this paper is as follows. After giving some general definitions, we introduce and study a hierarchical small-world network in Sec. III. In Sec. IV we study complete networks as another simple case. In Sec. V we present the results of our numerical simulations and scaling arguments for random small-world networks. Finally we give the conclusion remarks.

II. GENERAL DEFINITIONS

We take a one-dimensional chain with N nodes, numbered 1 to N . This chain has $N-1$ regular bonds, with spring constant k . The network structure is completed by adding some shortcuts of spring constant k' . In the following we will consider the case $0 \leq k' \leq k$ which is more reasonable in physical

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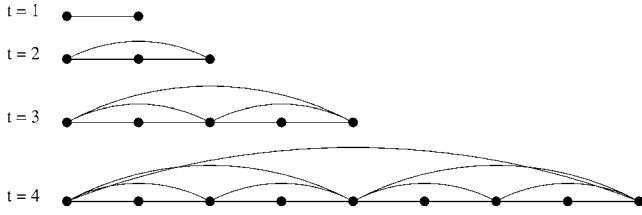


FIG. 1. Constructing a hierarchical small-world network.

models. The effective elastic constant of the network is denoted by K . We exert force F on the end nodes of a chain and obtain the extension distribution of shortcuts, $P(\Delta_x)$, where $k'\Delta_x$ is the force acting on a shortcut. On increasing F quasi-statically, we define a cutoff length δ_c for shortcuts. It means that a shortcut is torn if its extension exceeds δ_c . Here we will take $\delta_c=1$ (the specific value of δ_c does not affect the qualitative behavior of our results). For a given network, we may have a number of avalanches in some particular forces. An avalanche starts by the tearing of a shortcut and ends when all the shortcuts have an extension smaller than δ_c . By increasing F , a number of avalanches may occur which we label by index a . In each avalanche, one can measure, for example, the force value F_a , the chemical end-to-end distance R_a , and the effective spring constant of the network K_a . The CEED, R , is defined as the number of bonds in the shortest path connecting node 1 to N . Just after the a th avalanche, the physical end-to-end distance X_a is given by F_a/K_a . We also define the change in a quantity such as R by $\Delta R_a := R_{a+1} - R_a$.

III. A HIERARCHICAL SMALL-WORLD NETWORK

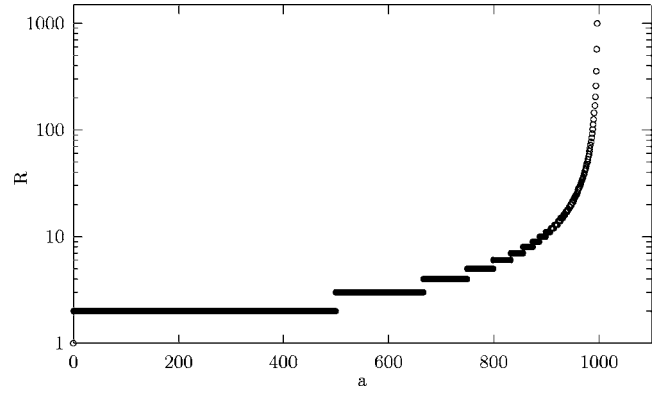
Let us first study a simple hierarchical small-world network where its behavior can be treated exactly. The construction of the network is depicted in Fig. 1. In the first step $t=1$ we consider two nodes connected with a regular bond. In the next step $t=2$ we make a copy of the previous step and merge them. We also add an additional shortcut between the first and last nodes of the new network. If we repeat this procedure for t steps we obtain a hierarchical SWN of size $N(t)=2^{t-1}+1$ with $M(t)=2^{t-1}-1$ shortcuts. One can obtain the following relation for the effective stiffness of the network:

$$K(t) = k' + \frac{K(t-1)}{2} = \frac{k}{N(t)-1} + 2k' \left(1 - \frac{1}{N(t)-1} \right). \quad (1)$$

If we pull the end nodes of the network with force F we find that there are 2^l shortcuts of the extension $\Delta_x = F/[K(t)2^l]$. Here $l \in [0, t-2]$ is an integer that labels the shortcuts according to the steps that have been added to the network. This in turn results in the following extension distribution:

$$P(\Delta_x) \sim (\Delta_x)^{-2}. \quad (2)$$

Now we start to increase F from zero. In very small F , $R_0=1$. When the force reaches to $F_1=K(t)\delta_c$, the spring connecting node 1 to $N(t)$ extends by δ_c and tears. After this event, $R_1=2$ and the number of shortcuts decreases to $M_1=M(t)-1$. Moreover, the effective network stiffness is


FIG. 2. Chemical end-to-end distance vs a for a complete network of size $N=1000$ with $k=1$ and $k'=0$.

given by $K_1=K(t-1)/2$. It is easy to see that a shortcut connecting two far nodes has a larger extension than one connecting two near nodes. Thus we can summarize the behavior of interesting quantities versus a

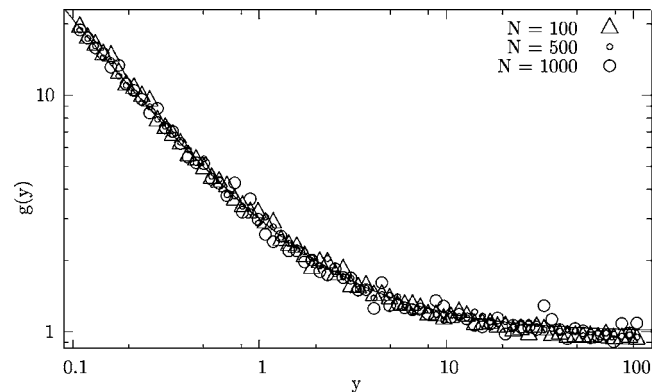
$$\begin{aligned} F_a &= K(t-a+1), \\ R_a &= 2^a, \\ M_a &= M(t) + 1 - 2^a, \end{aligned} \quad (3)$$

$$K_a = \frac{K(t-a)}{2^a}.$$

The physical end-to-end distance then reads

$$X_a = \frac{F_a}{K_a} = R_a \frac{K(t-a+1)}{K(t-a)}, \quad (4)$$

which for $t-a \gg 1$ is approximately equal to R_a . Thus, in this limit the physical end-to-end distance is well given by CEED. From the above relations we obtain $F_a=K_{a-1}R_{a-1}$. Consider the bonds of the shortest path connecting node 1 to N . F_a is the force at which the extensions of these bonds become δ_c . Note that, however, the relation between force and CEED is linear


FIG. 3. $g(y)=2K/(kpN)$ vs $y=pN^2$ for networks of size $N=100, 500$ (averaged over 1000 realizations) and $N=1000$ (averaged over 300 realizations). The line represents the curve $1+2/y$.

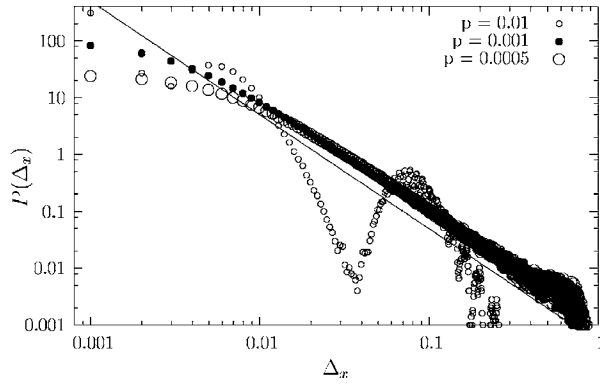


FIG. 4. Extension distribution for RSWNs of size $N=1000$ and $k=k'=1$ averaged over 5000 realizations. The line displays a power law of exponent -2 .

$$F_a = \frac{kR_a}{2^t} + 2k' \left(1 - \frac{R_a}{2^t}\right). \quad (5)$$

From Eqs. (3) one easily finds that

$$\begin{aligned} \Delta R_a &= -\Delta M_a = 2^a, \\ \Delta F_a &= \frac{k}{2^{t-a}} \left(1 - \frac{2k'}{k}\right). \end{aligned} \quad (6)$$

We see that for $k' < k/2$, F_a increases versus a . Thus in this situation we have a continuous change in R . In the continuum approximation we obtain

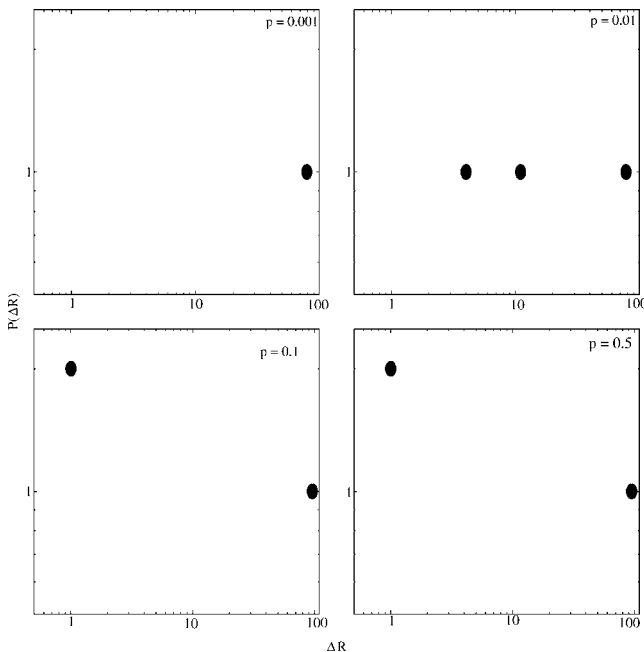


FIG. 5. $P(\Delta R)$ for single realizations with different p when $k=k'=1$, $N=100$. There are a few small changes in R followed by a large change of order N happened at critical force F_c .

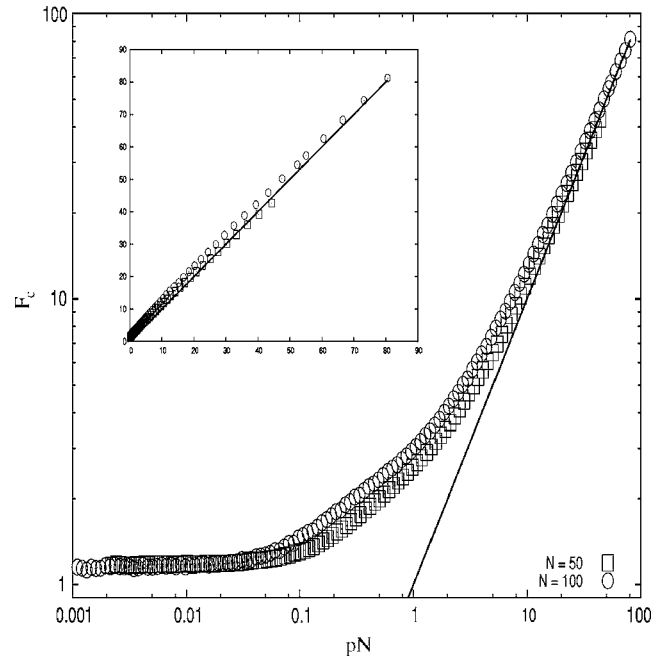


FIG. 6. F_c vs pN in RSWNs with $k=k'=1$ for $N=50$ and $N=100$ averaged over 5000 and 1000 realizations, respectively. The solid line displays $F_c = kpN$. In each step of the quasistatic process we increase F by 0.1. The inset shows more clearly the linear behavior of F_c for large pN .

$$P(\Delta R) \sim (\Delta R)^{-1}. \quad (7)$$

On the other hand, when $k' > k/2$, from Eqs. (3) and (6) we have $F_1 = K(t)$ and $F_1 > F_2 > F_3 \dots$. It means that after tearing the first shortcut, we reach a new configuration in which the extension of most extended shortcut(s) is(are) greater than δ_c . This process continues until all shortcuts tear. Note that during this event which occurs at $F = F_1$, CEED changes from 1 to $N-1$. Thus we have an abrupt change of CEED at $F_c = K(t)$.

IV. COMPLETE NETWORKS

In a complete network, each node is connected to all the other nodes. Thus the number of shortcuts is $M_0 = (N-1)(N-2)/2$. We start with the simple case of $k'=k$ with the effec-

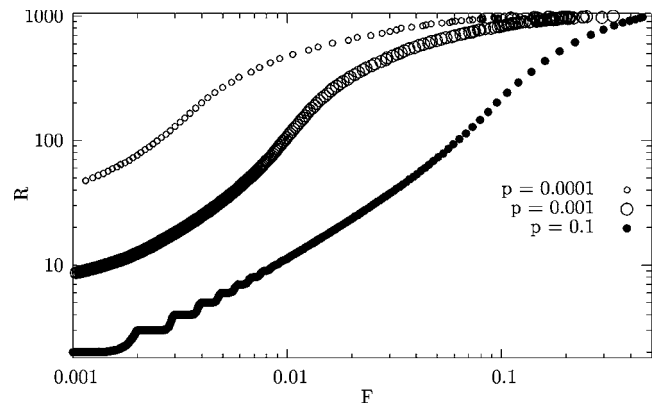


FIG. 7. The CEED vs F in RSWNs of size $N=1000$, $k'=0$ and $k=1$ after 5000 (for $p=0.1, 0.001$) and 10000 (for $p=0.0001$) realizations.

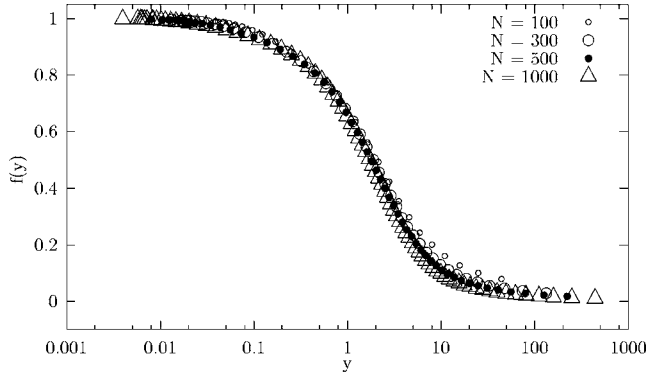


FIG. 8. $f(y)=R/N$ vs $y=pL_F^2$ for RSWNs of different sizes (averaged over 10 000 realizations).

tive network stiffness of $K_0=kN/2$. Moreover, $P(\Delta_x)$ consists of three peaks corresponding to three different kinds of shortcuts in the network: The first kind is a single shortcut between the end nodes and has the largest extension. The second kind of shortcuts which have less extensions, are those that connect the end nodes to the inner nodes. And the remaining shortcuts, with zero extensions, are the shortcuts connecting the inner nodes to each other.

In the absence of external force, $R_0=1$. For a given F we have the largest extension in the bond connecting node 1 to N . Hence by increasing F the first avalanche occurs at $F_1=kN/2$. At this step we have $M_1=M_0-1, R_1=2$, and $K_1=k(N-2)/2$. A simple calculation shows that for all springs connected to nodes 1 and N , we have $\Delta_x=F/[k(N-2)]$. So by increasing F we reach $F_2=k(N-2)$ in which $2(N-3)$ shortcuts are teared. In this step, we will have a complete network consisted of all nodes between 2 to $N-1$. But we know already that the spring connecting node 2 to $N-1$ can only stand forces lower than $k(N-2)/2$, which is much smaller than F_2 . Thus, in the second avalanche, all shortcuts will be removed. So $F_c=k(N-2)$ is the threshold F in which CEED has an abrupt change.

Next we consider the simple case of $k'=0$. It is clear that in this situation, the shortcuts have no contribution in the elastic properties of the network and we have a trivial problem in this respect. But in the process of increasing F , the

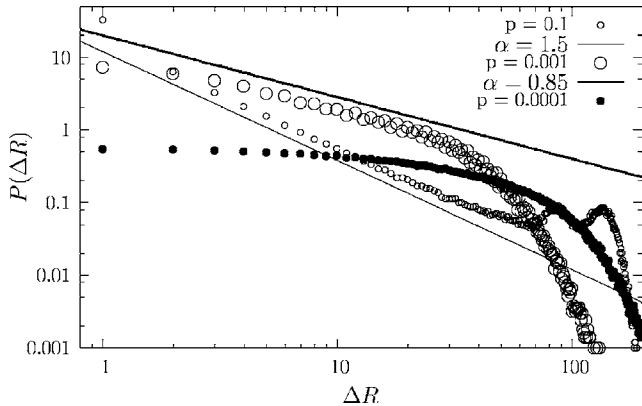


FIG. 9. $P(\Delta R)$ for RSWNs of size $N=1000, k'=0$ and $k=1$. The lines display $P(\Delta R) \propto (\Delta R)^{-\alpha}$. Number of realizations are the same as those of Fig. 6.

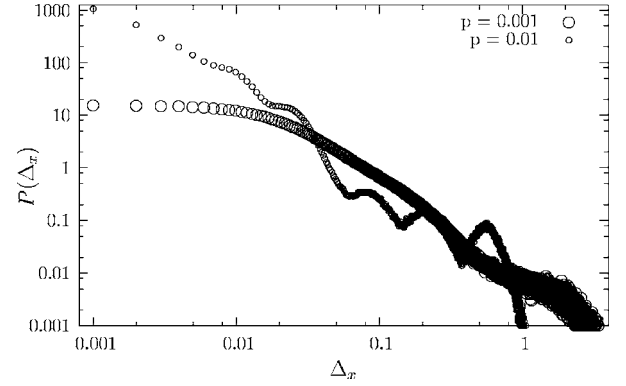


FIG. 10. Extension distribution for RSWNs of size $N=1000, k=1$, and $k'=0.1$ averaged over 5000 realizations.

effect of shortcuts in quantities like CEED is still important and nontrivial. It is not difficult to show that in this limit $P(\Delta_x) \propto (FN/k - \Delta_x)$. Moreover, as a function of the number of avalanches a , we have

$$F_a = \frac{k}{N-a},$$

$$R_a = 1 + \left[\frac{N}{N-a} \right]_+,$$

$$M_a = M_0 - \frac{a(a+1)}{2},$$

(8)

$$K_a = \frac{k}{N-1},$$

where $[x]_+$ denotes the smallest integer larger than or equal to x . In Fig. 2 we have shown R versus a . Here again the relation between force and CEED is linear. If we take a and R_a as continuous variables and use Eq. (8) we obtain

$$P(\Delta R) \sim (\Delta R)^{-3/2}.$$

(9)

This behavior is also seen in mean field models of fiber bundles for the size distribution of avalanches [18].

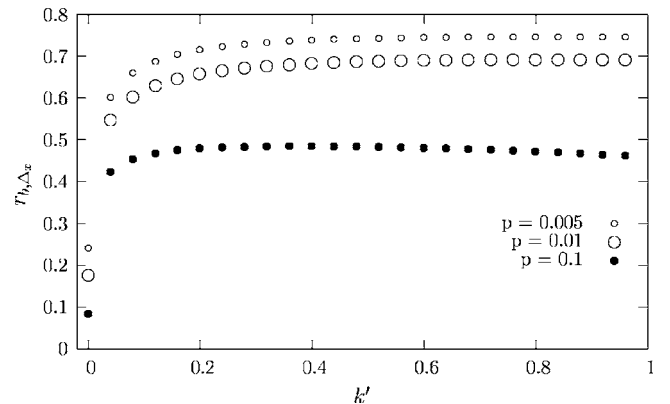


FIG. 11. Correlation coefficient of Δx and b for $N=100$ after averaging over 5000 realizations.

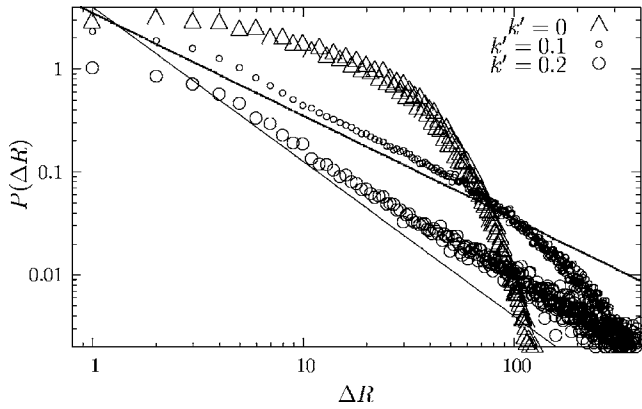


FIG. 12. $P(\Delta R)$ for a network with $N=1000, p=0.001$, and $k=1$. The data are results of averaging over 10 000 ($k'=0$) and 5000 ($k'=0.1, 0.2$) realizations. The lines represent power laws of exponent -1 and $-\frac{3}{2}$.

For $0 < k' < k$, $P(\Delta_x)$ still consists of three delta peaks. Numerical simulations show that only for $k' \ll k$ the abrupt change of CEED is replaced by a continuous one (more precisely a staircaselike behavior). For example, if we choose $N=100$ and $k=1$ we obtain a nearly continuous transition only for $k' < 0.01$. We found that this value of k' is a decreasing function of network size. Thus for $N \rightarrow \infty$ a continuous transition occurs only at $k'=0$.

V. RANDOM SMALL-WORLD NETWORKS

We take a one-dimensional chain of N nodes with $N-1$ regular bonds. Then with probability p , we connect any two nodes of distance larger than 1 by a shortcut. In the following, we consider three different cases (i) $k=k'$, (ii) $k'=0$, and (iii) $0 < k' < k$.

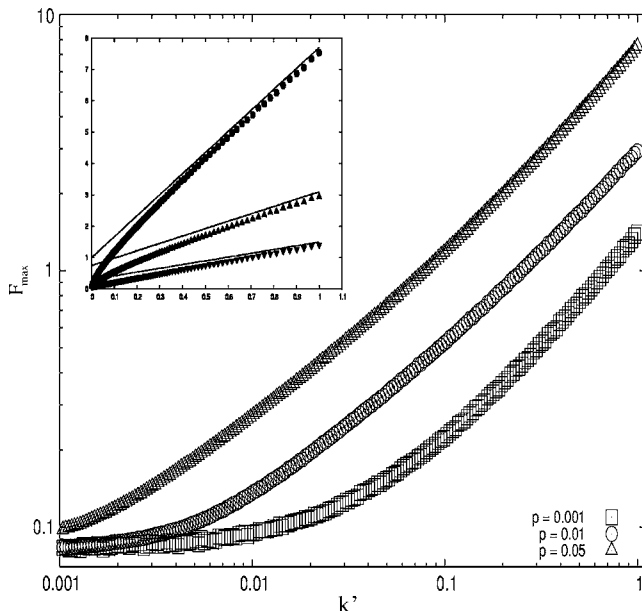


FIG. 13. F_{max} vs k' for $N=100$ and $k=1$ (averaged over 1000 realizations). The inset shows more clearly the linear behavior of data for large k' . The lines display linear functions $F_{max} = a_0 + a_1 k'$.

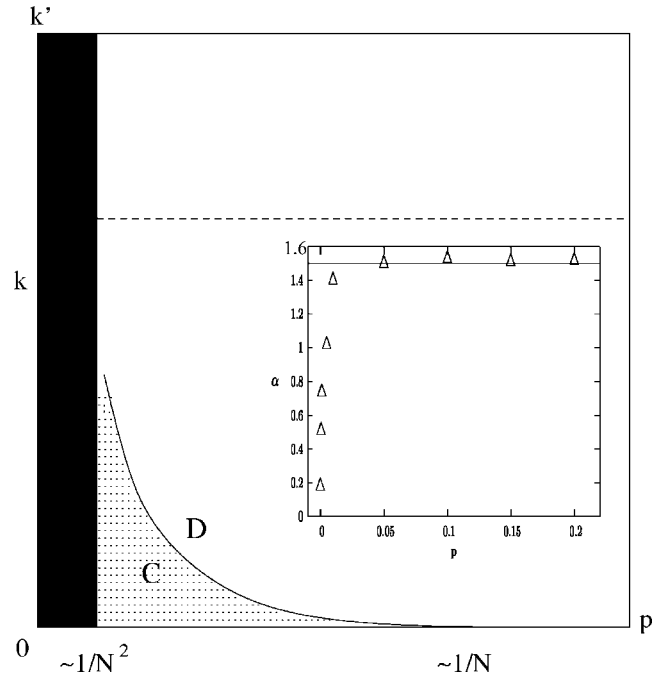


FIG. 14. Schematic representation of phase diagram. Dashed region indicates where we have a continuous behavior of R vs F . For $p < 1/N^2$ (black region) we cannot speak of continuous or discontinuous behavior of R . Note that as N approaches to infinity the continuous region reduces to the line $k'=0$. The inset shows the variation of α with p when $N=1000, k=1$, and $k'=0$. The errors in α are of order 0.01. The same behavior will be observed for α when we fix p and increase k' . The solid line in the inset indicates the expected value of α for large p , i.e., $\frac{3}{2}$.

A. The case $k=k'$

For $p > 1/N$ we expect to have $K = kpN/2$ which in the limit of $p=1$ is equal to the stiffness of a complete network. In other words if $p > 1/N$, we have effectively a complete network in which k has been replaced with kp . On the other hand for $p=0$ and large N we have $K = k/N$. Using these limiting cases we suggest that

$$\frac{K}{N} = \frac{kp}{2} g(pN^2),$$

$$g(y) \sim \begin{cases} 1, & y \gg 1 \\ \frac{2}{y}, & y \rightarrow 0 \end{cases} \quad (10)$$

Note that for large N , $pN^2/2$ is equal to the average number of shortcuts. Numerical simulations shown in Fig. 3 support this scaling relation.

As p decreases, we observe a crossover in $P(\Delta_x)$ from a three-modal behavior to a scale free one, see Fig. 4. When $p > 1/N$ this distribution has three broad maximums instead of three peaks in a complete network. The broadening of these peaks is due to the random structure of the SWN. Just in the small-world regime, $P(\Delta_x)$ is given by a power law distribution of power -2 . In this respect the RSWN belongs to the universality class of the hierarchical model introduced in Sec. III.

TABLE I. Summary of results for $P(\Delta_x)$ and $P(\Delta R)$ in different cases. Here $k'_c(p)$ gives the line in the (p, k') plane that separates continuous and discontinuous behavior of CEED.

	$P(\Delta_x)$	$P(\Delta R)$
$k=k'$	$(\Delta_x)^{-2}$	$\delta_{\Delta R, N}$ (an abrupt change of R)
$k'=0$	$a_0 - a_1 \Delta_x$	$(\Delta R)^{-\alpha} (\alpha \leq 3/2)$
$0 < k' < k$	$(\Delta_x)^{-\beta} (\beta \leq 2) +$ oscillations (for large p)	$(\Delta R)^{-\alpha} (\alpha \leq 3/2), k' < k'_c(p)$ $\delta_{\Delta R, N}, k' > k'_c(p)$

In our quasistatic process of increasing F , CEED changes nearly abruptly at F_c . This behavior has been shown in Fig. 5 that displays $P(\Delta R)$ for single realizations of the process. We observe that after a few events of small size, we have a large change of order N in R which we interpret it as a discontinuous behavior of CEED. Again for $p \gg 1/N, F_c = kp(N-2)$, see Fig. 6. For $p \rightarrow 0$, when there is at least one shortcut in the network, $F_c \approx k\delta_c$. These limiting behaviors of F_c with p have been shown in Fig. 6. We find that average number of shortcuts emanating from a node, pN , determines F_c . As far as the shortcuts do not overlap with each other, F_c does not depend on their number. Actually for a finite number of shortcuts, a typical configuration has almost nonoverlapping shortcuts. In this situation we have $F_c \approx k\delta_c$. Thus, we expect that F_c changes considerably only when $pN \sim 1$. Moreover, from the linear feature of the system we also expect that F_c should be proportional to δ_c and k . Thus we suggest $F_c = k\delta_c h(pN)$, where $h(pN)$ is a dimensionless scaling function. In Fig. 6 we show that this relation works well for two small-size networks and within our numerical errors.

B. The case $k'=0$

When $k'=0$ the shortcuts have no contribution in the elastic properties of the network. As indicated in the study of complete networks, $P(\Delta_x)$ is a linear decreasing function. But notice that during the quasistatic process, tearing of shortcuts leads to considerable variations in R . In Fig. 7 we show variation of R versus F obtained by numerical simulations. There is no linear relation between R and F . Indeed for a given F , all shortcuts connecting two nodes of distances larger than $L_F = k/F$ have been already teared. This introduces another length scale. We expect R to be a function of the number of shortcuts in a subnetwork of size L_F , that is,

$$\frac{R}{N} = f(pL_F^2). \quad (11)$$

Numerical simulations presented in Fig. 8 confirm this scaling relation.

Figure 9 displays $P(\Delta R)$ for various values of p . For a small value of $p=0.1, P(\Delta R) \propto (\Delta R)^{-3/2}$ like a complete network. For smaller p , we find a power law distribution of lower exponent. For a very small $p, P(\Delta R)$ is nearly a constant function with an exponential tail.

C. The case $0 < k' < k$

As before we take $k=1$ and decrease k' from one. Numerical simulations show that in this case the network stiff-

ness is proportional to k' . The effects of k' and p on elastic properties are the same. It means that with respect to elastic properties, the decreasing of k' at a fixed p is equivalent to the decreasing of p at a fixed k' . In Fig. 10 we display the extension distribution when $k'=0.1$. It is observed that when $p > 1/N$ we have a multimodal distribution in contrast to the three-modal distribution of case $k=k'$ for the same value of p . Recall that we did not have such a behavior for lower values of p when $k=k'$. Indeed this new multimodal behavior is observed only when we have a large number of shortcuts.

The extension distribution of RSWNs shows that there are some shortcuts bearing very large forces compared with the other ones. Certainly this is because of their essential roles in the network structure. A good measure of centrality of bonds in a network is their betweenness [2]. Suppose that we have n shortest paths connecting node 1 to N . A given shortcut may contribute in n_s of these paths. Then, the betweenness of this shortcut is defined as $b = n_s/n$. We define

$$r_{b, \Delta_x} := \frac{\langle b \Delta_x \rangle - \langle b \rangle \langle \Delta_x \rangle}{\sqrt{\sigma_b \sigma_{\Delta_x}}}, \quad (12)$$

as a measure of correlation between extension and betweenness of shortcuts. In this definition σ_{Δ_x} and σ_b are variances of Δ_x and b , respectively. In Fig. 11 we show how the correlation coefficient depends on k' and p . As expected, r_{b, Δ_x} has a considerable positive value in the small-world regime. The correlation coefficient is nearly independent of k' except for a rapid decrease to zero for $k' \rightarrow 0$.

In the quasistatic process, as it happens in complete networks, a considerable number of avalanches takes place only for a significantly small k' . Indeed, the larger p the lower value of k' ; we need to see a continuous behavior of R . In Fig. 12 we compare $P(\Delta R)$ for two cases of $k'=0$ and $k'=0.1$. The figure shows that, as expected, by increasing $k', P(\Delta R)$ approaches to that of a complete network. Indeed, in the small-world regime and for a nonzero $k', P(\Delta R)$ obeys a power law of exponent $-\alpha$ with $\alpha < \frac{3}{2}$. This exponent approaches to $-\frac{3}{2}$ as we increase k' or p .

In this case it is not easy to define $k'_c(p)$, the value which separates continuous and discontinuous regimes. Instead we calculate F_{max} , the force which leads to $R_{max} = N-1$. We expect that a change in the behavior of F_{max} with respect to k' signals a crossover from a continuous to a discontinuous region. In Fig. 13 we show the variation of F_{max} versus k' obtained by numerical simulation. We see that by decreasing k' , the linear behavior of F_{max} changes and then it saturates

for $k' \rightarrow 0$. This crossover occurs for a lower k' as one enhances the number of shortcuts.

VI. CONCLUSIONS

We studied the static and quasistatic properties of SWNs. We observed that the network's structure significantly affects the static and quasistatic behavior. Thus, we could draw some conclusions about the structure of networks through the study of these properties.

The summary of the main results for RSWNs have been represented in Table I and Fig. 14. We found that for very small values of p , the effective stiffness of the network is comparable with that of a complete network. When $k=k'$ the extension distribution of shortcuts is a power law of exponent -2 . In this respect RSWNs behave like a hierarchical network introduced in this paper. There was also a strong positive correlation between the betweenness of a shortcut

and its extension. It means that just by looking at the distribution of extensions in a network, one could be able to distinguish which shortcuts are more central.

In the quasistatic part, we showed that by increasing F , CEED could have a continuous or discontinuous transition. In general, to have a continuous transition we need a much smaller spring constant for shortcuts rather than regular bonds. In the case of a discontinuous transition, the critical force is determined by the average number of shortcuts per node. It was found that for a continuous transition, $P(\Delta R)$ is given by power law distributions of the exponent $-\alpha$ with $\alpha \leq \frac{3}{2}$.

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